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Intermediate / Information

Kleiber's Law Applied to Link Population Size With City Growth Aspects

Kleibers's Law is a biological law, which states that an organism's mass to the three-fourths power is proportional to its basal metabolic rate. I compared characteristics of Kleiber's law with how growth aspects of a city scale with that city's population using computer programming, and found that Kleiber's Law applies to model city growth, in some cases with higher certainty than in others.

Project Forms

Kleiber's Law Applied to Link Population Size With City Growth Aspects

Motivation:

Due to my passion for mathematics and computers, I knew that I wanted to pick a project that I would enjoy working on and studying. I was also interested in the implications and uses of a project. While reading a book about inventions, titled *Where Good Ideas Come From: The Natural History of Innovation* [12], I read a chapter that talked about Kleiber's Law and how it could potentially be used to model city growth. My interest was immediate, and I knew I wanted to better understand the topic.

Background Research:

Kleiber's Law is a law named after Max Kleiber, a Swiss agricultural biologist, who in the 1930's discovered a relationship between the mass and the metabolic rate of organisms. This law states that an organism's mass (M) to the three-fourths power (3/4) is proportional to its metabolic rate (q_0), or amount of energy per unit time that an organism needs to power its body while at rest [Equation 1]. For example, this means that an animal having 1000 times the mass of another would have around 178 times the metabolic rate, not 1000 times. The power constant of 3/4 came from the original estimate of 2/3; however, it was changed through empirical testing since it was found to be more accurate. The 3/4 constant may come from the fractal nature of organisms, which adds an extra dimension to both the numerator and the denominator of the exponent as opposed to the 2/3, which is based on Euclidean Geometry [11]. The exponent may also be explained from the fact that the blood flow of a certain organism is proportional to the 1/12 power, which when added to 2/3 results in 3/4. [3]

$$q_0 \propto M^{3/4} \quad \textbf{(1)}$$

Relationship between mass (M) and the metabolic rate (q_0) of an organism

$$y = ax^b \quad (2)$$

Underlying power law followed by Kleiber's Law.

Problem Statement:

I identified the relationship between the population and the growth factors (GDP, land area, infrastructure, etc.) of a city, and then established a parallel with Kleiber's Law for organisms. I was also faced with the challenge of finding real world applications for the modification of Kleiber's Law.

Hypothesis:

The relationship between a city's population and some growth factors follows a power law similar to Kleiber's Law. It was postulated that GDP and Housing Units would follow a power function well, but other growth aspects like land area would not. This was thought because GDP and Housing Units rely on the social interactions made in the city, but land area is usually a constant independent of what goes on in the city. The relationship was presumed to be much stronger for these factors, meaning that their predicted power function model would have a lower error, or less deviation to the real value than other aspects.

Procedure:

Take a city and analyze growth aspects of that city (e.g. GDP, infrastructure, land area, etc.) with respect to that city's population. Mathematica's nonlinear regression was used to fit my data to the power model equation of the form $y=ax^b$ [Equation 2]. To more easily compare my output (the predicted power function returned) to the actual data points, and the results seen in a data set mimicking Kleiber's Law I plotted the results in a log-log coordinate system because a power law is linearized thus. I graphed my data points and my predicted power function in log-log coordinates with sigma confidence bands, which show what percentage of the data points are predicted to fall between two same coloured bands, with increasing z-scores (standard deviations). I also performed statistical testing and was able to display an ANOVA (Analysis of Variance) table showing the standard error, t-stat, p-value, and separately the R^2 coefficient of correlation showing how closely the data points match the predicted line (on a scale of 0 to 1). Using computer programming, mainly Mathematica, and public sources such as the 2010 U.S. Census Bureau [2], I was able to gather data for this analysis.

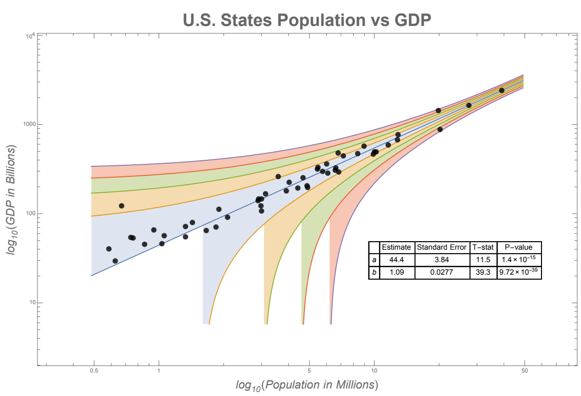

Figure 1. (Left) The figure (Source: [16]) shows the predicted power function line in log-log coordinates (blue line), the data points (as black dots), the sigma bands of confidence (in alternating colors beginning around the predicted power function), and the ANOVA table (on the bottom right).

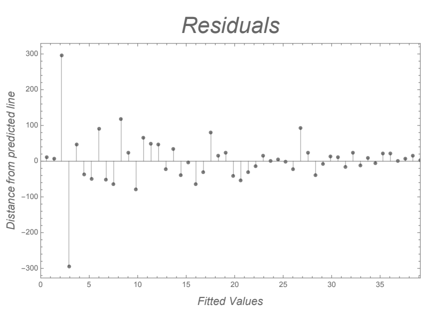

Figure 2. (Right) This density distribution shows how a given data set is distributed, and where the highest density of data points lie. (Lighter colours show higher densities).

Figure 3. (Above) This residual graph shows the distance between my predicted line and each data point in the actual data set.

Results or Observations:

The results of my analysis were positive and followed what I had predicted in my hypothesis. I believed that the results of some of the different growth aspects of a city would follow a power function similar to the one present in Kleiber's Law. Given the data sets I tested for, I was able to confirm my hypothesis. I quantified the similarities by finding the R^2 coefficient of correlation for each data set, which tells how accurately the data points fit the predicted model, and found that for the data sets I tested, there were high coefficients of correlation (close to 1), telling me that the power function model was a good fit for that given data set.

50 states: {0.983545, 0.982873}
18 cities: {0.582652, 0.530483}
Metabolic rate: {0.654172, 0.652958}
125 top cities: {0.721416, 0.716886}
Housing: {0.995827, 0.995794}
'10: {0.995718, 0.995543}
'11: {0.99581, 0.995639}
'12: {0.995878, 0.995709}
'13: {0.995908, 0.995741}
'14: {0.995985, 0.995821}
Average: {0.787522, 0.775799}

Figure 4. (Right) This list shows the R^2 coefficient of correlation and the R^2 coefficient of correlation adjusted by Mathematica for the different data sets tested, and the average of the first five unique data sets.

Conclusions:

When observing cities and their respective growth aspects throughout different data sets, the model Kleiber's Law follows is also followed by the data sets. I can support this claim with the results of my data, which returned a predicted power function, and an R^2 coefficient of determination, which tells how well the data points fit that given power function [Figure 4]. Much of the relevance of these conclusions has to do with real world applications. Given cities' populations, it is now possible to determine a great deal about the growth aspects of those cities by simply passing their respective populations into a predictive power function such as the one used in this analysis. The results from that analysis may be used to better focus resources, which would then allow a considerable savings of capital.

Real world applications:

This project can affect the real world today. Using a simple model, given only a city's population, endless information may be determined. This has implications for government to know exactly where to spend money; for urban planners to know about infrastructure and building patterns; for environmentalists, to counteract pollution; for ecologists to show the relationships between organisms; and of course for biologists to model relationships such as metabolic rate, heart beat rate, life span, brain mass, etc. Kleiber's Law may also be modified to model financial situations such as the relationship between a company's employee population and that company's profits, or it may also be applied to a government and its long term urban planning. The government would predict the following year's population for their city and pass that into the model. Based on the results of that model, they could anticipate the expected increase in housing units, which would then determine how many houses should be build to accommodate the growing population. By quantifying the number of resources to be put into a specific area, they are being utilized more effectively.

Future work:

If I were to further develop this project, I would be interested in testing many more sets of data, including data sets in a variety of areas such as finance. I would then develop my prediction models to add complexity and accuracy, and I would work on different ways of representing my results.

Sources of Error:

Sources of Error may be present in the original data sets (the population of cities, etc.); however, there is a low likelihood of this occurring at least for population surveys since the U.S. Census Bureau is quite reliable in its counts. The occasional human error may also be present.

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Bibliography/References:

[1] Annual Estimates of the Resident Population: April 1, 2010 to July 1, 2015 Source: U.S. Census Bureau, Population Division Release Date: December 2015

[2] Annual Estimates of Housing Units for the United States, Regions, Divisions, States, and Counties: April 1, 2010 to July 1, 2014 Source: U.S. Census Bureau, Population Division Release Date: May 2015

[3] Banavar, Jayanth R. et al. “A General Basis for Quarter-Power Scaling in Animals.” Proceedings of the National Academy of Sciences of the United States of America 107.36 (2010): 15816–15820. PMC. Web. 16 Apr. 2016.

[4] Bettencourt, Luís MA. "The origins of scaling in cities." science 340.6139 (2013): 1438-1441.

[5] "BioMath:Allometry." BioMath:Allometry. N.p., n.d. Web. 04 Mar. 2016.

[6] "Changing the Speed of Exported AVI Videos." Export. N.p., n.d. Web. 04 Mar. 2016.

[7] Chen, Yanguang. "An allometric scaling relation based on logistic growth of cities." Chaos, Solitons & Fractals 65 (December 2013): 65-77.

[8] "City Mayors: Largest Cities in the World and Their Mayors - 1 to 150." City Mayors: Largest Cities in the World and Their Mayors - 1 to 150. N.p., n.d. Web. 07 Apr. 2016.

[9] Cottineau, C., Hatna, E., Arcaute, E., and Batty, M. (2015). Paradoxical interpretations of urban scaling laws.

[10] Geoffrey West. " The surprising math of cities and corporations" TED. July 2011. Lecture.

[11] J. Zhao, A common origin for 3/4- and 2/3-power rules in metabolic scaling (2015).

[12] Johnson, Steven. Where Good Ideas Come From: The Natural History of Innovation. New York: Riverhead, 2010. Print.

[13] "How Do I Plot Coordinates (latitude and Longitude Pairs) on a Geographic Map?" Plotting. N.p., n.d. Web. 04 Mar. 2016.

[14] Oliveira, Erneson A., José S. Andrade, and Hernán A. Makse. "Large cities are less green." Scientific Reports 4 (February 2014).

[15] Peterson, Garry. "Faculty of 1000 Evaluation for Growth, Innovation, Scaling, and the pace of Life in Cities." F1000 - Post-publication Peer Review of the Biomedical Literature (n.d.): n. pag. Web.

[16] "Regional Economic Accounts." U.S. Bureau of Economic Analysis (BEA). N.p., n.d. Web. 24 Apr. 2016.

[17] St. ESPRC Workshop Scaling in Social Systems (n.d.): n. pag. Web.